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Generalization of the BLM procedure and its scales in any order of pQCD.

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Abstract: The Brodsky–Lepage–Mackenzie procedure is sequentially and unambiguously extended to any fixed order of perturbative QCD beyond the so called "large–β_0 approximation". As a result of this procedure, the obtained perturbation series looks like a continued-fraction representation. A subsequent generalization of this procedure is developed, in order to optimize the convergence of the final series, along the lines of the Fastest Convergence Prescription. This generalized BLM procedure is applied to the Adler D function and also to $R_{e^+e^-}$ in QCD at $\mathrm{N^3LO}$. A further extension of the sequential BLM is presented which makes use of additional parameters to optimize the convergence of the power-series at any fixed order of expansion.

Keywords: QCD, Renormalization Group.

Contents

1. Introduction

The first goal of this article is to extend the well-known Brodsky, Lepage and Mackenzie (BLM) procedure [1] of scale setting to any fixed order of perturbative QCD (pQCD). We will show that higher orders of pQCD in the $\overline{\rm MS}$–scheme unambiguously determine the new scales in the sequentially extended BLM prescription. More specifically, the effects of the coupling renormalization, encoded in the β-function coefficients, are absorbed into a set of proper scales μ_i^2 of the couplings $a_i \equiv \alpha_s(\mu_i^2)/(4\pi)$ at any fixed order of pQCD. Thus, in contrast to the prevailing opinion, presented in the Particle Data Group booklet [2] (see rev. "Quantum Chromodynamics" therein), higher-order corrections can indeed "fix" the scales for the sequentially extended BLM (seBLM) procedure. At the same time, it may happen that this unambiguous seBLM choice of $a_i,\ \mu_i^2$ is not the "best" one from the point of view of the best series convergence, so that one should treat these two phenomena separately. For this reason, our first goal includes the analysis of the final seBLM result with an explicit demonstration of the above-mentioned problem. Our second goal following from this problem is to supply the seBLM procedure with a mechanism resembling the Fast Apparent Convergence (FAC) [3] (but not coinciding with it) in order to improve the convergence of the series.

To simplify the analysis of the structure of the radiative corrections, renormalization-group invariant quantities, like the Adler function (D), are considered below. Quantities like the heavy quark potential V_Q [4], the Bjorken sum rules, the Gross-Llewellin Smith sum rule and so on, see, for example, Ref. [6, 5] for a review, can be considered in the same manner. Here an empirical relation between the QCD β–function coefficients b_i, $b_i \sim b_0^{i+1}$ has been used that will be explained later. The hierarchy of the contributions of the coupling renormalization to the perturbative coefficients at every order of $a_s \equiv \alpha_s(\mu^2)/(4\pi)$ is based on this power law for b_i. This detailed hierarchy requires a matrix representation for the perturbation-series expansion (PE) rather than the standard series one. The mentioned power relation works well at least up to the currently known coefficient b_3 and $N_f = 0 \div 5$ active flavors in contrast to the usually discussed proposition of the so-called "large–b_0" limit that really implies $-N_f \gg 1$ (therefore, $b_0 \sim b_1$ in that approximation), see for a review [7]. The seBLM procedure is based on this matrix representation; therefore, it can be formulated in terms of the dynamical characteristics, namely, the β–function coefficients, rather than in terms of certain $\mathrm{SU}_c(3)$–Casimir operators that may appear at an intermediate stage. The use of the relation $b_i \sim b_0^{i+1}$ allows one to go beyond this large–b_0 approximation in a natural way. Note that this power law should, of course, fail at some higher order of the PE when its expected factorial divergence starts to become operative. The proposed coupling renormalization is worked out (Sec. 5 and 6) and includes as a particular case the "bubble approximation" elaborated on in [8, 9, 10]. The corresponding new perturbation series obtained with the seBLM procedure is also worked out. Then the β–function expansion is performed explicitly for the 4-loop D-function and, subsequently, the seBLM procedure is applied to this D-function to highlight the advantages and disadvantages of the procedure in the case of this physical quantity. Although higher-order expansion coefficients have been calculated for other observables, mentioned above, the calculation of the beta-function expansion for these quantities constitutes a different task, and asks for a careful treatment. For this reason, we restrict ourselves here to the consideration of the D and R functions.

Different kinds of extensions of the BLM approach have been discussed in a number of interesting articles [8, 9, 10, 12, 5, 11, 13, 14] which appeared in the last decade. The important issues about the scheme ambiguity of the BLM procedure [15], as well as the role of the anomalous dimensions (for the corresponding quantities) in the optimization procedures [16], will not be discussed here, with the corresponding calculations being performed in the $\overline{\mathrm{MS}}$–scheme for massless QCD. The present approach differs from those works mainly in three items:
(i) it is based on a detailed matrix presentation for the PE;
(ii) the proposed generalized scheme is formulated explicitly for any fixed order of the PE;
(iii) all the sources of the coupling renormalization are taken into account and absorbed into the coupling scales.

The seBLM procedure may not necessarily pertain to an improvement of the perturbation series. A relevant example was mentioned even in the pioneering work of Ref. [1]. An improved machinery, which uses the proper scales of the seBLM, applying a FAC-inspired optimization, is considered in Sec. 8. This generalization of the seBLM optimizes the perturbation series and formally completes the developed method in the sense of convergence. Because this method employs the x-portions of the proper scales of the seBLM procedure in order to achieve the best convergence, it is named here xBLM procedure.

This work is outlined as follows. In the next section, a convolution representation is proposed, which can be useful for the interpretation of the BLM procedure. This interpretation naturally leads to a certain kind of the "skeleton expansion" and is presented in a complete form in Appendix C. Appendix A, referred to in the same section, contains the proof of the convolution representation and the necessary formulas for the β-function coefficients. Section 3 describes in detail the standard BLM procedure and

defines the task, while Sec. 4 presents the matrix rearrangement of the perturbation-series expansion. The first stage of the sequential generalization to a fixed N-order case is considered in Sec. 5. The remaining steps and the completion of the task are given in Sec. 6, with some technical details presented for the convenience of the reader in App. B. An application of the developed method to the Adler function is considered in Sec. 7, with relevant technical details relegated to App. D. Finally, Sec. 8 presents the optimized version of our approach that yields a better convergence of the perturbation-series expansion.

2. Convolution representation for amplitudes

Here we rewrite the standard perturbation power-expansion series for an amplitude in the form of a formal integral representation. This representation, in contrast to similar ones in [9, 8] and in [14], does not involve integration over the intrinsic momentum k. Even more, it is not related to a Feynman integral over the momenta at all. The properties of this representation will be discussed in this section and we shall use it in the next section as a convenient "perturbative tool" to interpret the BLM procedure by means of the average virtuality flow.

Let us consider the formal perturbation series $s(a_s)$ for the two-point amplitudes as functions of the external momentum Q^2. The coupling a_s is normalized for the default choice of the scale $\mu^2 = Q^2$. In this case, the coefficients of the expansion, d_n, are numbers in MS–like schemes due to the cancellation of the logarithms $\ln^k(Q^2/\mu^2)$, but they leave behind remnants: the constant parts ($\ln(C)$) of these logarithms (accompanied by the β-function coefficients) are left over in the d_n coefficients and have to be taken care of. For further convenience, we introduce a new expansion parameter $A = b_0 a_s$ "normalized" by the factor b_0 so that

$$s(a_s) = d_0 + \sum_{n=1} a_s^n d_n \equiv S(A) = d_0 + \frac{d_1}{b_0} \cdot \sum_{n=1} A^n D_n \quad \text{with} \quad D_1 = 1 \tag{2.1}$$

and define new coefficients $D_i = \dfrac{d_i}{d_1 b_0^{i-1}}$ that will simplify the intermediate calculations, and will help us maintain the contact with the "large b_0" limit, $b_0 \gg 1$ at $A \lesssim 1$. Note that in the real QCD theory, and below the c-quark threshold (for $N_f = 3$), we have at $\mu^2 = 1$ GeV^2 the values $b_0 = 9 \gg 1$ and $A(\mu^2) \equiv \alpha_s(\mu^2)\dfrac{b_0}{4\pi} \approx 0.32 < 1$ at the NLO level. The running of the coupling $A \to \bar{A}(t)$ (or $a_s \to \bar{a}_s(t)$) follows the renormalization group (RG) equation

$$\frac{d}{dt}\bar{A} \equiv B(\bar{A}) = -\left(\bar{A}^2 + c_1\bar{A}^3 + c_2\bar{A}^4 + \ldots\right) \tag{2.2}$$

$$\text{with } c_i = \frac{b_i}{b_0^{1+i}}, \tag{2.3}$$

where $B(A)$ is the modified β–function and $t = \ln\left(Q^2/\Lambda^2\right)$ is a natural variable for MS-like schemes. At the one loop order we have a well-known solution to (2.2): $\bar{A}_{(1)}(t) = \dfrac{1}{t}$; at the two-loop order the exact solution $\bar{A}_{(2)}$ can be realized in terms of the appropriate branch of the Lambert function W_{-1}, see the discussion in Appendix A, and [17, 18] for details. Beyond the two-loop level, explicit solutions are difficult to obtain. The loop order l is indicated by the subscript l in parenthesis, e.g., $A_{(l)}$.

To evaluate the sum in (2.1), we need an appropriate representation for the coefficients D_n. To this end, let us decompose these coefficients into two parts

$$D_n = D^+(n) + (-1)^{(n-1)} D^-(n) \tag{2.4}$$

to distinguish the values of the coefficient for even and odd values of the index n. Then, we construct the generating functions $\mathcal{P}^+$, $\mathcal{P}^-$, respectively, for these parts $D^+(n)$, $D^-(n)$ in the following way:

$$D^{\pm}(n) = \int_0^\infty \mathcal{P}^{\pm}(\alpha)\alpha^{n-1}d\alpha \tag{2.5}$$

with the normalization condition

$$D_1 = D^+(1) + D^-(1) \equiv \int_0^\infty \left(\mathcal{P}^+(\alpha) + \mathcal{P}^-(\alpha)\right) d\alpha = 1. \tag{2.6}$$

At large α the behavior $\mathcal{P}^{\pm}(\alpha) \sim \alpha^{\gamma+1}e^{-\alpha/c}$ corresponds to the expected asymptotic behavior of the expansion coefficients D_n [19] at large n

$$\mathcal{P}^{\pm}(\alpha) \sim \alpha^{\gamma+1}e^{-\alpha/c} \to D^{\pm}(n) \sim D_n \sim \Gamma(n+1)\ n^\gamma c^n, \tag{2.7}$$

which is a familiar behavior exhibiting a purely renormalon divergence proportional to $n!$. Below, we shall construct a representation for the sum (2.1), based on Eq. (2.2) and on the representation (2.5).

1-loop integral representation. At 1-loop level, evolution (2.2) leads to a useful representation for the powers of $\bar{A}_{(1)}$, viz.,

$$\frac{1}{n!}\left(-\frac{d}{dt}\right)^n \bar{A}_{(1)} = \left(\bar{A}_{(1)}\right)^{n+1}. \tag{2.8}$$

Substituting (2.5) and (2.8) into definition (2.1) and *changing* the order of the sum and the integration, we obtain the formal integral representation

$$S(\bar{A}_{(1)}) = d_0 + \frac{d_1}{b_0}\cdot\int_0^\infty \left\{\mathcal{P}^+(\alpha)\left[\exp\left(-\alpha\frac{d}{dt}\right)\bar{A}_{(1)}(t)\right] + \mathcal{P}^-(\alpha)\left[\exp\left(\alpha\frac{d}{dt}\right)\bar{A}_{(1)}(t)\right]\right\} d\alpha\ ,$$

which contains the shift-argument operators. Finally, we find the representation

$$S(\bar{A}_{(1)}) = d_0 + \frac{d_1}{b_0}\cdot\int_0^\infty \left\{\mathcal{P}^+(\alpha)\bar{A}_{(1)}(t-\alpha) + \mathcal{P}^-(\alpha)\bar{A}_{(1)}(t+\alpha)\right\} d\alpha \tag{2.9}$$

that is *linearized* in the coupling $\bar{A}_1$ in terms of the convolution

$$\int_0^\infty \left(\mathcal{P}^+(\alpha)\bar{A}_{(1)}(t-\alpha) + \mathcal{P}^-(\alpha)\bar{A}_{(1)}(t+\alpha)\right) d\alpha \equiv \langle\bar{A}_{(1)}(t-\alpha)\rangle + \langle\bar{A}_{(1)}(t+\alpha)\rangle. \tag{2.10}$$

Following Neubert's proposal [8], it is convenient to consider the integration in Eqs. (2.9) and (2.10) as an average of the corresponding coupling A over $\mathcal{P}$. This is denoted as an average of the coupling $A(t \mp \alpha)$ on the right-hand-side (RHS) of (2.10).

Representation (2.9) seems to be close to that invented by Neubert [8], and also by Beneke, Braun, and Ball [9, 10], though it turns out to be different. The main difference between the two representations will be demonstrated in the next subsection. Note that representation (2.9) is a "formal" one because the change of the order of summation and integration in its derivation has not been proved. Nevertheless, the integration over the Taylor expansion of the factors $\bar{A}_{(1)}(t-\alpha)$ and $\bar{A}_{(1)}(t+\alpha)$ in the integrand of Eq. (2.9) leads again to the initial series, given by Eq. (2.1), up to any finite order of the expansion. Therefore, we shall call this Eq. (2.9) the "sum" of the perturbative series (2.1) in the one-loop running-coupling approximation. The integrand in Eq. (2.9) has a pole singularity at $t = \alpha$, which reflects the fact that this representation is ill-defined. One can now discuss how the contour should be deformed to give a rigorous sense to this integral [20, 21]. The residue of the pole can be taken as a measure for the uncertainty of

the asymptotic series and it reflects the factorial growth of the perturbation-series coefficients D_n (see [8, 7, 20]).

We are not going to discuss here the non-perturbative interpretation of Eq. (2.9). Let us only mention that quantum field models with an integrable running coupling $\bar{A}_{(1)}(t)$ lead to a finite expression for the convolution given by Eq. (2.10). Therefore, the finiteness of the coupling of the analytic perturbation theory (APT) [23, 24], $A^{\rm APT}_{(1)}(t) \leq 1$ provides an example for a convergent perturbation series in the sense of the convolution representation (i.e., in a weak sense). For a brief discussion of this feature see Appendix A.

***l*-loop generalization of the integral representation**. Let $\bar{A}_{(l)}$ be the solution of the RG equation in the l-loop approximation on the RHS of Eq. (2.2). Then, the initial series, Eq. (2.1), with the coupling $\bar{A}_{(l)}$ can be represented as

$$S(\bar{A}_{(l)}) = d_0 + \frac{d_1}{b_0} \cdot \left\{ \int_0^\infty \mathcal{P}^+(\alpha)\ \bar{A}_{(l)} \left(t + \alpha \cdot \frac{\bar{A}^2_{(l)}}{B(\bar{A}_{(l)})} \right) d\alpha \ + (\pm \to \mp) \right\} \tag{2.11}$$

that should be compared with Eq. (2.9). For example, for the two-loop running of $\bar{A}$, the representation given by Eq. (2.10) for the average coupling, should be replaced by

$$\langle \bar{A}_{(2)}(t \mp \frac{\alpha}{1 + c_1 \bar{A}_{(2)}(t)}) \rangle = \int_0^\infty \mathcal{P}^\pm(\alpha) \bar{A}_{(2)} \left(t \mp \frac{\alpha}{1 + c_1 \bar{A}_{(2)}(t)} \right) d\alpha \,. \tag{2.12}$$

Representation (2.11) can be proved in the same way as Eq. (2.9) in the one-loop case (see Appendix A). Its Taylor expansion in the second term of the argument of $\bar{A}_{(l)}\,(t \mp \ldots)$ generates the PE in Eq. (2.1) for $A = \bar{A}_{(l)}$. In what follows we shall call $S(\bar{A}_{(l)})$ in Eq. (2.11) the "sum" of the perturbative series in the l-loop running-coupling approximation.

3. An Illustration: 1-loop BLM procedure

Here we consider the standard BLM procedure from the $\mathcal{P}$-distribution point of view. First, we rewrite the series, Eq. (2.1), in the form of the average

$$S(\bar{A}_{(1)}) = d_0 + \frac{d_1}{b_0} \cdot \langle \bar{A}_{(1)}(t - \alpha) \rangle \,. \tag{3.1}$$

Note here that in the one-loop approximation one can take only one of its parts for simplicity and then use the linear property of this average. Expanding the coupling in the average $\langle \bar{A}_{(1)}(t - \alpha) \rangle$ around t_0 and taking into account $\bar{A}(t_0) = A_0$, one finds

$$\langle \bar{A}\,((t - t_0 - \alpha) + t_0) \rangle = A_0 - A_0^2(t - t_0 - \langle \alpha \rangle) + A_0^3 \left((t - t_0)^2 - 2(t - t_0)\langle \alpha \rangle + \langle \alpha^2 \rangle \right) + \ldots . \tag{3.2}$$

In order to fix an appropriate scale t_0, it is necessary to demand that the NLO coefficient (at the order A_0^2) in the expansion (3.2) or some particular part of it (to be defined later) should be nullified. To this end, let us consider first the structure of $\langle \alpha \rangle \equiv D_2$ in detail.

D_2 is proportional to the coefficient d_2 of the initial series (2.1) which contains a term proportional to b_0 appearing due to the coupling renormalization. In the discussion to follow, it turns out to be convenient to use the following shorthand notation:

$$d_2 = d_1 \cdot (b_0\ e + f) \equiv d_1 \cdot \left((b_0)^1\ d_2[1] + (b_0)^0\ d_2[0] \right) . \tag{3.3}$$

Here and below, the square brackets contain the powers of the β-function coefficient b_0 associated with the expansion coefficient d_2. The coefficient $d_2[1]$ originates from the diagrams shown in Fig. 1(a), (b), where the ellipses denote other diagrams with the vertex and the gluon or quark field renormalization. The coefficient $d_2[0]$ stems mainly from the diagrams displayed in Fig. 1(c), (d), this coefficient is not responsible for the coupling renormalization. The notation $d_2[\ldots]$ in (3.3) is particularly useful in the multiloop case because it allows the inclusion of the powers of the higher β-function coefficients. Then, we get

$$\langle\alpha\rangle \equiv D_2 = \frac{d_2}{d_1 b_0} = d_2[1] + \frac{d_2[0]}{b_0}. \tag{3.4}$$

Two possibilities to fix t_0 are considered in the literature:

(*i*) $t_0 = t - D_2 \Rightarrow \mu^2 = Q^2 \exp(-D_2)$ which means that this coefficient is nullified in Eq. (3.2). This content of the coefficient D_2 is included as a whole via the new expansion parameter A_0. This is the meaning of the Fastest Apparent Convergence (FAC) procedure [3] denoted by

$$\langle \bar{A}_{(1)}(t-\alpha)\rangle = A_0 + A_0^2 \cdot 0 + O(A_0^3). \tag{3.5}$$

Here the structure of the $O(A_0^3)$ tail looks like $A_0^3\left(D_3 - D_2^2\right) + O(A_0^4)$ [8]. The distribution $\mathcal{P}$ in this approximation can be reduced to $P_{\rm FAC}$, $\mathcal{P}^{\pm} \to P^{\pm}_{\rm FAC(1)} = \theta(\pm D_2)\ \delta(\alpha - |D_2|)$.

(*ii*) Alternatively, we may demand that the contribution $d_2[1]$ to the coefficient D_2, which is $\propto b_0$, is nullified so that $t_0 = t - d_2[1] \Rightarrow \mu^2 = Q^2 \exp(-d_2[1])$. In this case, only the term $d_2[1]$ responsible for the coupling renormalization is included by the new expansion parameter A_0. This defines the BLM procedure [1, 5]

$$\langle \bar{A}_{(1)}(t-\alpha)\rangle = A_0 + A_0^2 \cdot \frac{d_2[0]}{b_0} + O(A_0^3). \tag{3.6}$$

The remaining term of this procedure proportional to A_0^2 is suppressed by the inverse power of the "large b_0". The N^{2}LO term in Eq. (3.6) looks like $A_0^3\left(D_3 - 2D_2 d_2[1] + (d_2[1])^2\right)$. The distribution $\mathcal{P}$ in the BLM case can be reduced to $P_{\rm BLM}$, $\mathcal{P}^{\pm} \to P^{\pm}_{\rm BLM(1)} = \theta(\pm d_2[1])\ \delta(\alpha - |d_2[1]|)$. This determines an intrinsic "scale" $d_2[1]$ and shifts the normalization scale from t to t_0.

Let us consider now the generating functions $\mathcal{P}$ from the point of view of their standard PE. In this case, $\mathcal{P}$ can be represented as a formal series; viz.,

$$\mathcal{P}^{\pm}(\alpha) = \sum_{n=0} \frac{(\mp 1)^n}{n!} \delta^{\{n\}}(\alpha) \cdot D^{\pm}(n+1) \tag{3.7}$$

that generates just the standard PE, $\sum_{n=1} A^n D_n$, in Eq. (2.1). In practice, only a few terms of this series are known. Such a broken series in (2.1) corresponds to the broken series in (3.7) where every term is strongly concentrated around the origin $\alpha = 0$. However, in the case of a smooth function $\mathcal{P}$, that is not strongly concentrated near the origin, (3.7) provides only an insufficient approximation to $\mathcal{P}$. Indeed,

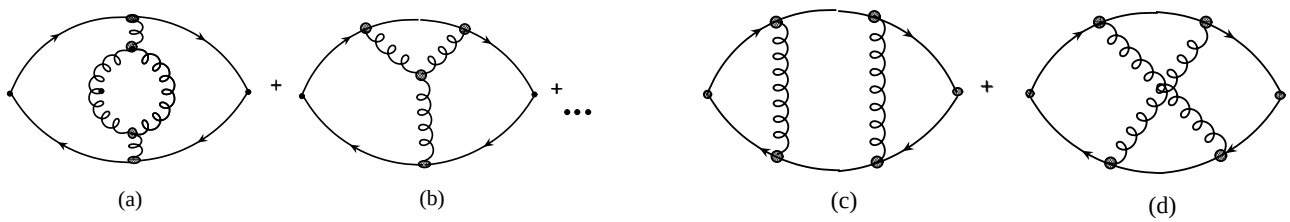

Figure 1: Diagrams (a,b,...) contribute to $b_0\ d_2[1]$; diagrams (c,d) contribute to $d_2[0]$

one should take into account much more terms of the expansion in Eq. (3.7) or even its infinite subseries to approximate the real behavior anywhere but not close to the origin, say, near the first extremum of the $\mathcal{P}(\alpha)$ in α. The first two terms involved into the BLM procedure

$$\delta(\alpha) - \delta^{\{1\}}(\alpha) d_2[1] + \dots$$

generate an infinite model sum, like (at $d_2[1] > 0$)

$$\mathcal{P}^+(\alpha) \to P^+_{\mathrm{BLM}(1)}(\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta^{\{n\}}(\alpha) \left(d_2[1]\right)^n = \delta(\alpha - d_2[1]) \tag{3.8}$$

in contrast to the standard PE in Eq. (3.7) with only a few known terms. The model distribution $P^+_{\mathrm{BLM}(1)} = \delta(\alpha - d_2[1])$ looks preferable because this approximation takes into account at once (admittedly, in a rather crude manner) the main feature of behavior of $\mathcal{P}(\alpha)$ near the extremum at $\alpha \approx d_2[1]$. What parts of the PE in the next higher orders should one involve in the procedure to improve P_{BLM}? Or, stated differently, which are the diagram classes that generate these parts? To clarify this issue (and to define the next approximation of $\mathcal{P}$), one should analyze the structure of the d_n coefficients at N^{n}LO of the PE. This is the task of the next section.

4. Perturbation expansion – from series to matrix

In the previous section, we considered the structure of the coefficient d_2, Eq. (3.3), which is the basis of the standard BLM. Now, let us extend this representation to the next higher coefficient appealing to the evident diagrammatic origins of the renormalizations there. In N^{2}LO, see examples of the diagram elements in Fig. 2(a,b), the renormalization of a_s coming from one–gluon exchanges and from vertices, and quark field renormalization of the same order of expansion generates contributions proportional to $a^3 b_0^2$, $a^3 b_1$. These terms should be compared with those originating from Fig. 1(a) and Fig. 1(b) in NLO. A new kind of contribution like $a^3 b_0$ is generated by the two-gluon exchange with the renormalization of one of these gluon lines/vertices, see Fig. 2(c). The representation for d_3 is similar to that in Eq. (3.3) and looks like an expansion in a power series in $b_0, b_1, \dots$, i.e.,

$$d_3 = d_1 \cdot \left(b_0^2 d_3[2,0] + b_1 d_3[0,1] + b_0 d_3[1,0] + d_3[0,0] \right), \tag{4.1}$$

where the first argument n_0 of the expansion coefficients $d_n[n_0, n_1, \dots]$ corresponds to the power of b_0, whereas the second one n_1 corresponds to the power of b_1, *etc.* The coefficient $d_n[0,0]$ represents so–called [15] "genuine" corrections with $n_i = 0$ for all possible b_i powers. One typical diagram generating such contributions is depicted in Fig. 2(e) (compare with the diagrams in Fig. 1(c) and Fig. 1(d)). If all the arguments of the coefficient $d_n[\dots, m, 0, \dots, 0]$ to the right of the index m are equal to zero, then we shall omit these arguments for simplicity and write instead the brief notation $d_n[\dots, m]$.

In N^{3}LO the renormalization of a_s generates contributions proportional to $a_s^4 b_0^3$, $a_s^4 b_0 b_1$, $a_s^4 b_2$ originating from one–gluon exchanges and/or vertices. On the other hand, contributions proportional to $a_s^4 b_0^2$, $a_s^4 b_1$, $a_s^4 b_0$ originate from the mixing of the coupling renormalization stemming from different gluon lines and/or vertices. Finally, contributions like a_s^4 appear from "genuine" corrections, i.e., those corrections that come from diagrams which do not contain gluon or quark or intrinsic vertex renormalizations. Such diagrams do not generate contributions containing powers of the beta-function coefficients. Following these lines of argument, the d_4 coefficient looks in this brief notation like

$$d_4 = d_1 \cdot \left(b_0^3 d_4[3] + b_1 b_0 d_4[1,1] + b_2 d_4[0,0,1] + b_0^2 d_4[2] + b_1 d_4[0,1] + b_0 d_4[1] + d_4[0] \right). \tag{4.2}$$

The same ordering of the β-function elements holds for all higher coefficients d_n. It is convenient for our purposes to present this β-function structure in terms of the "normalized" variables $\bar{A}$ and D_n. The D_n coefficients have an evident form (in the left column the corresponding coupling powers $\bar{A}^n$ are shown)

$$
\begin{aligned}
&\bar{A}^1(t) \quad D_1 = \underline{1};\\
&\bar{A}^2(t) \quad D_2 = \underline{d_2[1]} + \frac{1}{b_0}\cdot d_2[0];\\
&\bar{A}^3(t) \quad D_3 = \underline{d_3[2]} + \underline{c_1 d_3[0,1]} + \frac{1}{b_0}\cdot\left(d_3[1] + \frac{1}{b_0}d_3[0]\right);\\
&\bar{A}^4(t) \quad D_4 = \underline{d_4[3]} + \underline{c_1 d_4[1,1]} + \underline{c_2 d_4[0,0,1]} +\\
&\qquad\qquad \frac{1}{b_0}\cdot\left(d_4[2] + c_1 d_4[0,1] + \frac{1}{b_0}\cdot\left(d_4[1] + \frac{1}{b_0}d_4[0]\right)\right);\\
&\bar{A}^5(t) \quad D_5 = \underline{d_5[4]} + \underline{c_1 d_5[2,1]} + \underline{c_1^2 d_5[0,2]} + \underline{c_2 d_5[1,0,1]} + \underline{c_3 d_5[0,0,0,1]} +\\
&\qquad \frac{1}{b_0}\cdot\Bigg(d_5[3] + c_1 d_5[1,1] + c_2 d_5[0,0,1] +\\
&\qquad\qquad \frac{1}{b_0}\cdot\left(d_5[2] + c_1 d_5[0,1] + \frac{1}{b_0}\cdot\left(d_5[1] + \frac{1}{b_0}d_5[0]\right)\right)\Bigg);\\
&\bar{A}^N(t) \quad D_N = \underline{d_N[N-1]} + \ldots
\end{aligned}
\tag{4.3}
$$

where c_i are defined in Eq. (2.3).

Here, we do not discuss how to derive the elements of this representation explicitly for the known multi-loop results. It is a separate task that is solved for the partial case of the Adler function in Appendix D. Note in this context that similar representations for some physically interesting cases were presented in [16, 25]. In the considerations to follow, we assume that the elements determining the coefficients D_n in Eq. (4.3) are known so that they can be used for further considerations. To clarify the meaning of the structure of the above representation, let us comment on the nature of its elements. The first column of the coefficients $d_n[n-1]$ in Eq. (4.3) corresponds to the "bubble approximation" that includes the contributions from the diagrams with the maximum number of bubbles in this order, see Fig. 1(a). These "bubble" contributions involved in the extended BLM procedure, were considered before in [8, 9, 10, 5]. However, there are other unsuppressed contributions in Eq. (4.3), which have been underlined for a faster identification. In fact, for the $\overline{\rm MS}$- scheme the values of the coefficients c_i, namely,

$$
c_1 \approx 0.79; \quad c_2 \approx 0.88; \quad c_3 \approx 1.9; \quad c_4 = \text{unknown} \tag{4.4}
$$

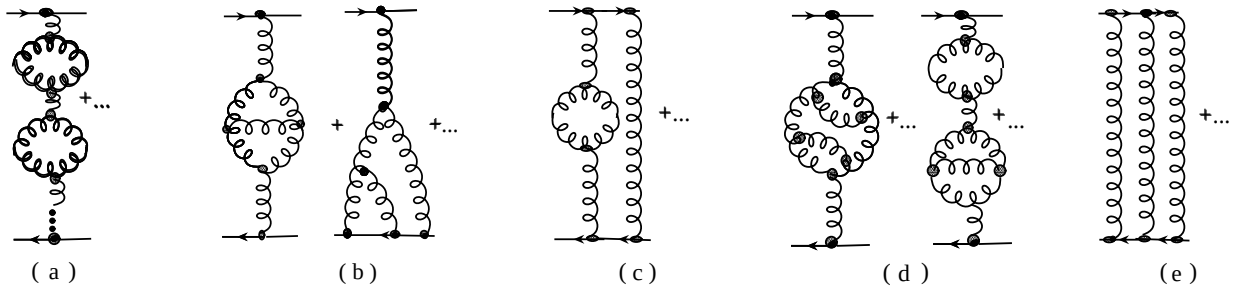

Figure 2: Diagram elements for α_s–radiative corrections contributing, e.g., to the photon polarization operator Π: (a) a chain of gluon bubbles contributing to the b_0^2-term in the coefficient d_3. The ellipses here denote other diagrams with the gluon/vertex renormalization; (b) diagrams contributing to the b_1–term in d_3 to illustrate the renormalization of gluon fields and vertices; (c) diagrams that contribute to the b_0–term in d_3; (d) diagrams generating b_2- and $b_0 b_1$- terms in d_4; (e) diagrams without any intrinsic renormalization contributions to $d_3[0]$.

are of order 1 where, for definiteness, their estimates have been obtained for $N_f = 3$ so that one has no reason to neglect them in Eq. (4.3). These terms (cf. the c_1-term in D_3 in Eq. (4.3)) originate in part from the diagrams in Fig. 2(b), while the c_2- and c_1- terms in D_4 in Eq. (4.3) stem from Fig. 2(d), and so on. To put our final results on a broader basis, one may imagine that the uncalculated coefficients c_i are also of the order of unity, $c_i = O(1)$, as we discussed in the Introduction.

We see that the PE series (2.1) subject to the representation (4.3) has a complicated structure. We face two different expansion parameters in Eq. (4.3): first, the coupling $\bar{A}$ for the lines and, second, b_0^{-1} for the columns. To simplify the treatment of these parameters, it is convenient to introduce the notation $\bar{A}^i \cdot y_{ij} \cdot b_0^{-j+1}$ for the contributions in Eq. (4.3) and $D_i = y_{ij} \cdot b_0^{-j+1}$ for their coefficients. $Y = \{y_{ij}\}$ is a triangular matrix with the elements $y_{nn} \equiv d_n[0]$ on the diagonal. These diagonal elements do not contribute to the coupling renormalization and are maximally suppressed by the b_0^{-1} powers in D_n, while the unsuppressed terms (underlined in Eq. (4.3)) are contained in the first column of Y

$$y_{11} \equiv \underline{1}; \tag{4.5}$$

$$y_{21} = \underline{d_2[1]}; \tag{4.6}$$

$$y_{31} = \underline{d_3[2] + c_1 d_3[0,1]}; \tag{4.7}$$

$$y_{41} = \underline{d_4[3] + c_1 d_4[1,1] + c_2 d_4[0,0,1]}; \tag{4.8}$$

$$\ldots = \ldots \qquad .$$

All these terms originate from the renormalization of a single coupling/gluon line. As an illustration, see the skeleton diagram in Fig. 3(a) which characterizes all diagrams like those in Fig. 2(a),(b),(d). Note that only the terms y_{i1} survive in the formal "large–b_0" limit, $b_0 \gg 1$ at $c_i = O(1)$.

5. The BLM task, first stage of the generalization

Here we construct the first stage of the generalization of the BLM procedure, basing it on the hierarchy of the contributions introduced in the previous section. We start with the most important unsuppressed contributions y_{i1}. The goal is to find a new pair $(t_1, A(t_1))$—following the RG law—which is able to nullify all the y_{i1} terms and then absorb them into the new expansion parameter $A(t_1)$:

$$\bar{A}(t) \xrightarrow{RG} \bar{A}(t_1) \equiv A_1; \quad t_1 \equiv t - \Delta_1$$
$$\Delta_1 \equiv \Delta_1(A_1) = \Delta_{1,0} + A_1 \cdot \Delta_{1,1} + A_1^2 \cdot \Delta_{1,2} + \ldots \tag{5.1}$$

The shift Δ_1 from the original scale t to the first intrinsic scale t_1 is going to be presented in the form of a perturbation series in A_1 (as first suggested in [12]). The corresponding procedure consists in re-expanding the initial expression $\bar{A}^i y_{ij} b_0^{-j+1}$ in terms of the new coupling A_1 and subsequent rearrangement of the power series. This includes the re-expansion of the coupling $\bar{A}(t) = A(\Delta_1, A_1)$ and its powers in terms of Δ_1,

$$\bar{A}(t) = A(\Delta_1, A_1) = A_1 - B(A_1)\frac{\Delta_1}{1!} + B'(A_1)B(A_1)\frac{\Delta_1^2}{2!} + \ldots$$
$$= \exp\left(-\Delta_1 B(\bar{A})\partial_{\bar{A}}\right) \bar{A} \mid_{\bar{A}=A_1}, \tag{5.2}$$

which can be realized with the operator $\exp\left(-\Delta_1 B(A)\partial_A\right)[\ldots] \mid_{A=A_1}$, see, e.g., [22]. Applying this expansion together with the expansion for Δ_1 in Eq. (5.1) to the initial expression $\bar{A}^i y_{ij} b_0^{-j+1}$, one arrives at the rearranged series in A_1, notably,

$$\exp\left(-\Delta_1(A_1) B(\bar{A})\partial_{\bar{A}}\right)\left[\bar{A}^i y_{ij} b_0^{-j+1}\right] \mid_{\bar{A}=A_1} = (A_1)^i \; y_{ij}^{(1)} b_0^{-j+1}. \tag{5.3}$$

The elements of the new matrix $y_{ij}^{(1)}$ are illustrated below in terms of the initial coefficients D_i and read

$$\bar{A}^1 \cdot D_1 \to (A_1)^1 \cdot 1; \tag{5.4}$$

$$\bar{A}^2 \cdot D_2 \to (A_1)^2 \cdot D_2 - 1\Delta_{1,0}; \tag{5.5}$$

$$\bar{A}^3 \cdot D_3 \to (A_1)^3 \cdot D_3 - 2\Delta_{1,0} \cdot D_2 - \Delta_{1,0}c_1 + \Delta_{1,0}^2 - \Delta_{1,1}; \tag{5.6}$$

$$\bar{A}^4 \cdot D_4 \to (A_1)^4 \cdot D_4 - 3\Delta_{1,0} \cdot D_3 + \left(3\Delta_{1,0}^2 - 2c_1\Delta_{1,0}\right) D_2 - c_2\Delta_{1,0} + \frac{5}{2}c_1\Delta_{1,0}^2 - \Delta_{1,0}^3 + \left(2\Delta_{1,0} - 2D_2 - c_1\right)\Delta_{1,1} - \Delta_{1,2}; \tag{5.7}$$

$$\bar{A}^N \cdot D_N \to (A_1)^N \cdot D_N - (N-1)\Delta_{1,0} \cdot D_{N-1} + \dots$$

The sequential BLM procedure requires that the $y_{i1}^{(1)}$ contributions *should cancel* at each order A_1^i and $i > 1$ in the set of Eqs. (5.5), (5.6), (5.7),... . This requirement completely determines the elements $\Delta_{1,i}$ of the expansion of Δ_1 in the original set of algebraic equations. Explicit expressions for the first few $\Delta_{1,k}$, and the omitted term in the equations below, are presented in Appendix B. At this point, we compile a few coefficients that are important for our discussion:

$$\Delta_{1,0} = y_{21} = d_2[1]; \tag{5.8}$$

$$\Delta_{1,1} = y_{31} - (y_{21})^2 - c_1 y_{21} = d_3[2] - d_2^2[1] + c_1\left(d_3[0,1] - d_2[1]\right); \tag{5.9}$$

$$\Delta_{1,2} = y_{41} - 3y_{31}y_{21} - 2(y_{21})^3 - c_1 \cdot \ldots = d_4[3] - 3d_2[1]d_3[2] + 2(d_2[1])^3 + c_1 \cdot \ldots \tag{5.10}$$

Note that the NLO BLM correction in Eq. (5.9) cancels in the particular case $d_3[2] = (d_2[1])^2$, $d_3[0,1] = d_2[1]$. The first of these terms corresponds to the geometric progression of the leading logarithms $d_n^{\rm BLM}[n-1] = (d_2[1])^{n-1}$, whereas the second one corresponds to the cancellation of sub-leading logarithms. If one applies these conditions to $\Delta_{1,2}$ in Eq. (5.10) (see also Eq. (B.4)), one obtains again the evident "geometric" condition $d_4[3] = (d_2[1])^3$ for the cancellation of the leading logarithmic part at the next step and so on. This test demonstrates the self-consistency of the calculations. If the coefficients $d_{n+1}[n]$ in y_{n1} follow the RG law for the leading logarithms, then they are taken into account already by the first element $\Delta_{1,0}$, while in order to take into account the sub-leading contributions, one needs more subtle conditions. The absorbtion of such contributions into the proper scales, see, e.g., the c_1-terms in Eq. (5.9), marks the differences between our procedure and other BLM extensions, for example, that of Ref. [5].

In this way, one can rearrange step by step for any fixed order N of the PE the first column y_{i1} in terms of Δ_1. As a result of this procedure, the initial series can be reduced to a new one that contains only one unsuppressed term $A_1 \cdot 1$, while all the others are suppressed by powers of b_0^{-1}, i.e.,

$$\sum_{i\geq j\geq 1} \left(\bar{A}\right)^i y_{ij} \cdot b_0^{-j+1} \stackrel{1\ \text{stage}}{\longrightarrow} \sum_{i\geq j\geq 1} (A_1)^i\, y_{ij}^{(1)} \cdot b_0^{-j+1} = A_1 + A_1 \cdot \sum_{i\geq j\geq 2} (A_1)^{i-1}\, y_{ij}^{(1)} \cdot b_0^{-j+1}. \tag{5.11}$$

In other words, the matrix Y transforms into a new matrix $Y^{(1)}$, the first column of which is now $y_{i\ 1}^{(1)} = \delta_{i\ 1}$ and the other few elements are presented in Appendix B, Eq. (B.1). These changes in the matrix Y (or in the coefficients D_i) are absorbed into the new normalization "scale" $t_1 = t - \Delta_1$ of the coupling $A_1 = \bar{A}(t_1)$. Let us stress that no approximation has been used in all steps of the procedure up to the A^N order. The above mentioned detailed hierarchy of the contributions is used only in order to establish the order of the sequential steps, rather than for performing an approximation. Indeed, we initiated this rearrangement with the first unsuppressed y_{i1} column, the most important part of the hierarchy. Now we will continue with the second column of $Y^{(1)}$, $y_{i2}^{(1)}$, that is suppressed by the power b_0^{-1} and proceed in similar manner with this column.

6. Sequential BLM procedure, next stages

Let us continue with the diagonalization of the matrix $Y^{(1)}$ by rearranging its second column. First, we single out the b_0^{-1}– suppressed terms on the RHS of Eq. (5.11) which are accumulated by its $(Y^{(1)})_{(n-1)}$– minor part in

$$\sum_{i\geq j\geq 2} (A_1)^{i-1}\, y_{ij}^{(1)} b_0^{-j+1}\,. \tag{6.1}$$

The elements of this minor are represented in the right part of Table 1 (put on the right of the double vertical line). The first power b_0^{-1}– suppressed terms are the first entries in each row of this column. Repeating the same procedure as at the first BLM stage (see the previous section) with the column $y_{n2}^{(1)}$,

Table 1: The structure of $A_1 \cdot (A_1)^{i-1}\, y_{ij}^{(1)} b_0^{-j+1}$, here $\tilde{y}_{n2}^{(1)} = y_{n2}^{(1)}/y_{22}$

$A_1\cdot$	1		minor elements
$A_1\cdot$	0+	$(A_1)^1\cdot$	$\dfrac{d_2[0]}{b_0}\cdot 1$
$A_1\cdot$	0+	$(A_1)^2\cdot$	$\dfrac{d_2[0]}{b_0}\cdot \tilde{y}_{32}^{(1)} + \dfrac{d_3[0]}{b_0^2}$
$A_1\cdot$	0+	$(A_1)^3\cdot$	$\dfrac{d_2[0]}{b_0}\cdot \tilde{y}_{42}^{(1)} + \dfrac{1}{b_0^2} y_{43}^{(1)} + \dfrac{d_4[0]}{b_0^3}$
$A_1\cdot$	0+	$(A_1)^4\cdot$	$\dfrac{d_2[0]}{b_0}\cdot \tilde{y}_{52}^{(1)} + \dfrac{1}{b_0^2} y_{53}^{(1)} + \dfrac{1}{b_0^3} y_{54}^{(1)} + \dfrac{d_5[0]}{b_0^4}$
$\dots$	0+	$\dots$	$\dots$

we rearrange again these terms with the help of the new expansion parameter $A(t_2)$ at the new scale t_2 to obtain

$$\begin{aligned} \bar{A}(t_1) \xrightarrow{RG} \bar{A}(t_2) &\equiv A_2; \\ t_1 - t_2 \equiv \Delta_2 &= \Delta_{2,0} + A_2\cdot\Delta_{2,1} + A_2^2\cdot\Delta_{2,2} + \dots\,, \end{aligned} \tag{6.2}$$

like in Eqs. (5.1), (5.3). As a result, all the terms from $y_{i2}^{(1)}$, except the first one, $y_{22}^{(1)} \equiv d_2[0]$, are transferred into the scale t_2 of the new coupling A_2—therefore, $y_{i2}^{(1)} \to y_{i2}^{(2)} = y_{22}\,\delta_{i2}$. Picking out this diagonal term on the RHS of (6.1), one arrives at the result of the second stage of the procedure

$$\sum_{i\geq j\geq 2}^{n} (A_1)^{i-1}\, y_{ij}^{(1)} b_0^{-j+1} \;\xrightarrow{2\ \text{stage}}\; A_2 \frac{d_2[0]}{b_0} + A_2 \sum_{i\geq j\geq 3}^{n} (A_2)^{i-1}\, y_{ij}^{(2)} b_0^{-j+1}. \tag{6.3}$$

To fix the new scale t_2 from $\Delta_{2,m}$ we use the same procedure as in the previous section. The first equalities in Eqs. (5.8)–(5.10) remain valid also for the elements $\Delta_{2,m}$ originating from $y_{i2}^{(1)}$ with an evident shifting of all its indices by 1. Therefore, using Eqs. (B.1) to determine $y_{i2}^{(1)}$ and choosing the common factor

$\frac{d_2[0]}{b_0}$ to normalize the elements ($\tilde{y}^{(1)}_{n2} = y^{(1)}_{n2}/y_{22}$), one arrives at

$$\Delta_{2,0} = \tilde{y}^{(1)}_{32} = \frac{y_{32}}{y_{22}} - 2y_{21} = \frac{d_3[1]}{d_2[0]} - 2d_2[1]; \tag{6.4}$$

$$\Delta_{2,1} = \tilde{y}^{(1)}_{42} - \left(\tilde{y}^{(1)}_{32}\right)^2 - c_1\tilde{y}^{(1)}_{32}; \tag{6.5}$$

$$\dots$$

Then substituting Eq. (6.3) into the RHS of Eq. (5.11) one obtains the final result of the second stage

$$\sum_{i\geq j\geq 1} \left(\bar{A}\right)^i y_{ij}\cdot b_0^{-j+1} \overset{\text{1. stage}}{\longrightarrow} \dots \overset{\text{2. stage}}{\longrightarrow} A_1\left(1 + A_2\left(\frac{d_2[0]}{b_0} + \sum_{i\geq j\geq 3}^{n} \left(A_2\right)^{i-1} y^{(2)}_{ij} b_0^{-j+1}\right)\right). \tag{6.6}$$

In this way, one can apply a chain of transformations $Y \to Y^{(1)} \to Y^{(2)} \to \dots \to Y^{(n-1)}$ from column to column to complete the diagonalization of the matrix Y, as it is displayed for the first stage in the left column of Table 1. At each stage of this diagonalization procedure, one will obtain a new coupling $A(t_i)$. The final result of this successive BLM procedure reduces the initial perturbation series, Eq. (2.1), to a series of the special form[1]

$$S(A) \to S_{\text{seBLM}} = d_0 + \frac{\bar{A}(t_1)}{b_0}\cdot d_1\left(1 + \frac{\bar{A}(t_2)}{b_0} d_2[0]\left(1 + \frac{\bar{A}(t_3)}{b_0} d_3[0]\,(1+\dots)\right)\right) \tag{6.7}$$

that contains only the "genuine" elements $d_i[0] = y_{ii}$, accompanied by the corresponding coupling at its proper scale t_i, $t_i = t - \Delta_1 - \dots - \Delta_i$. In other words, the seBLM procedure transforms the standard power series $\bar{a}^n_s(t)$ to the series of the products $\prod_{i=1}^n \bar{a}_s(t_i)$ keeping invariant the diagonal elements y_{ii}. These elements turn out to be the coefficients of the new series expansion.

Note that Eq. (6.7) can be easily presented in the form

$$S_{\text{seBLM}} = d_0 + \cfrac{a_1 d_1[0]}{1 - \cfrac{a_2 d_2[0]}{1 + a_2 d_2[0] - \cfrac{a_3 d_3[0]}{1 + a_3 d_3[0] - \cfrac{a_4 d_4[0]}{\dots}}}}, \tag{6.8}$$

where $a_i = \bar{a}_s(t_i)$, a form, motivated by the continued-fraction representation. The benefit of this representation is that it effects at every step the new coupling at its new appropriate scale. This equation itself may become the source of a new approximation for the initial series $S(A)$. An instructive interpretation of the procedure from the "distribution" point of view is presented in Appendix C. The diagrammatic sources for both the new expansion coefficients $d_i[0]$ and the new proper scales t_i are discussed there. In particular, the skeleton diagram with a single dressed-gluon connection (or vertices) in Fig. 3(a) generates the term $\bar{A}(t_1)/b_0\cdot d_1$ in Eq. (6.7), the shift of the scale Δ_1 is determined by the coupling renormalizations appearing from all corresponding partial diagrams, see Figs. 1, 2(a), and 2(b), and so on. Representation (6.7) corresponds to an expansion over the classes of skeleton diagrams and provides one with a particular model for $\mathcal{P}^+(\alpha)$.

The final result of the seBLM procedure is represented by Eq. (6.7) or Eq. (6.8). Let us reiterate that the underlying mechanism for obtaining this result is due to the rearrangement of the PE via the

[1] Let us stress here that the form of this series differs from those suggested in [5, 11].

Y-matrix, which also fixes the hierarchy of these coupling renormalization contributions. The constructed procedure takes into account *all* sources of the coupling renormalization appearing in the PE and absorbs them completely into the coupling scales, just in the spirit of the original BLM method [1]. At first glance, this looks as an improvement but it is rather formal because the seBLM procedure may result in a shifting of the scales into a region where the applicability of pQCD may not be justified for both $(A(t_i),\ t_i)$ and the new expansion coefficients $d_i[0]$. The question is under which conditions this procedure will provide us with a better convergent series. The standard BLM procedure may improve the NLO approximation, see Eq. (3.6), if both coefficients $d_2[1]$ and $d_2[0]$ have the same sign. Similarly, to the above the seBLM procedure may improve the convergence of the series when the elements of every line of the matrix Y have the same sign. In this case, the absolute magnitude of the final expansion coefficients $d_i[0]$ (see (6.7)) is less than that of the initial ones D_i. One, however, should not expect an improvement for the case when the elements in the lines of Y are alternating in sign and a strong cancellation appears in the coefficients $D_i = y_{ij} \cdot b_0^{-j+1}$ between different terms. We shall apply the seBLM procedure to the known N^{3}LO calculation of the D–function in Section 7 and argue that this is an example just for the latter case.

7. seBLM procedure for the D function

Here we consider step by step the results of the seBLM procedure to highlight its advantages and disadvantages from the point of view of the convergence of the PE of the Adler D-function.

The initial well-known series for D [26] can be rewritten by means of the β-function coefficients to read

$$\begin{aligned}
&D = 3\sum_f Q_f^2 \left\{d_0 + d_1\left[a + a^2 d_2 + a^3 d_3 + a^4 d_4 + \ldots\right]\right\},\ d_0 = 1;\ d_1 = 3C_F;\\
&d_2 = b_0 \cdot d_2[1] + d_2[0];\\
&d_3 = b_0^2 \cdot d_3[2] + b_1 \cdot d_3[0,1] + b_0 \cdot d_3[1] + d_3[0];\\
&d_4 = b_0^3 \cdot d_4[3] + b_0 b_1 \cdot d_4[1,1] + b_2 \cdot d_4[0,0,1] + b_0^2 \cdot d_4[2] + \ldots
\end{aligned} \tag{7.1}$$

However, to recast D into this form is a separate problem which is solved in Appendix D on the basis of the results obtained in [27] for the MSSM with light gluinos. This new degree of freedom allows us to distinguish the contributions with different coefficients b_i in d_3. Note that the expressions for the expansion elements in (7.1) remain valid even including light gluinos that contribute to the β-function (Appendix A). The explicit expressions for the elements $d_3[m,n]$ are presented in Appendix D, while below their numerical values are displayed and used

$$d_2 = b_0 \cdot 0.69\ +\ \frac{1}{3}; \tag{7.2}$$

$$d_3 = b_0^2 \cdot 3.104\ \ -\ \ b_1 \cdot 1.2\ \ \ + b_0 \cdot 55.70\ -\left(573.96 + 19.83\frac{(\sum_f Q_f)^2}{3(\sum_f Q_f^2)}\right). \tag{7.3}$$

For the sake of illustration, we substitute the value $b_0(N_f = 3) = 9$, $b_1(N_f = 3) = 64$ into (7.3),

$$d_3 = \quad 251.1\ \ -\ \ 76.8\ \ \ +\ \ 501.3\ \ \ -\ \ (573.96 + 0) \approx \underline{101.9}, \tag{7.4}$$

in order to compare the contributions stemming from different sources. Furthermore, we shall apply the seBLM procedure to D step by step to remove, respectively, the b_0-contribution to N^{2}LO, the b_0^2 and b_1-contributions to N^{3}LO, etc., and the associated results will be analyzed.

Now we start with our procedure with the original BLM scale setting by transforming the coefficients $d_2,\ d_3$ (cf. expressions (7.2)-(7.3)) and the coupling as follows

$$d_2 \to \tilde{d_2} = b_0 \cdot 0 + \frac{1}{3}; \tag{7.5}$$

$$d_3 \to \tilde{d_3} = b_0^2 \left(d_3[2] + d_3[0,1]c_1 - d_2^2[1] - d_2[1]c_1\right) + b_0(d_3[1] - 2d_2[0]d_2[1]) + d_3[0] \tag{7.6}$$

$$= b_0^2\ (\quad 2.1555 \quad - \quad 1.0251 \quad) + b_0(\ 55.70 - 0.46 \quad) + \ldots \approx \underline{14.7} \tag{7.7}$$

$$A(t) \to A(t_1);\ t - t_1 = \Delta_{1,0} = d_2[1] \approx 0.69. \tag{7.8}$$

We see that at the first step (employing the same condition as when we did with Eq. (7.4)) the value of $b_0^2\ y_{31}$ is reduced by approximately a factor of 2, whereas the value of $b_0\ y_{32}$ practically does not change. On the other hand, the value of the total coefficient is reduced to $\underline{14.7}$ as compared with the initial value $d_3 \approx \underline{101.9}$ in Eq. (7.4). This strong cancellation appears due to the large and negative value of the "genuine" term $d_3[0]$. The contents of d_4 in Eq. (7.1) also transform following Eq. (5.10). Appealing to the results provided in [28], which lead to $d_4[3] \approx 2.18$, one can predict the modification of the"bubble part" $d_4[3]$ of the d_4,

$$d_4[3] \approx 2.18 \to d_4[3] - d_3[2]d_2[1] - 2d_2[1](d_3[2] - d_2[1]^2) \approx -3.3\,,$$

which looks rather moderate.

As the next step of seBLM, the modified $\tilde{y}_{31}$ term in Eq. (7.6) is transferred into Δ_1, following Eqs. (5.8-5.9),

$$\tilde{d_3} \to \tilde{\tilde{d_3}} = b_0^2 \cdot 0 + b_1 \cdot 0 + b_0 \cdot (d_3[1] - 2d_2[0]d_2[1]) + d_3[0] \approx -\underline{77}; \tag{7.9}$$

$$A(t) \to A(t_1);\ t - t_1 = \Delta_1 = d_2[1] + A(t_1) \cdot \left(d_3[2] + d_3[0,1]c_1 - d_2^2[1] - d_2[1]c_1\right); \tag{7.10}$$

$$\Delta_1 \approx 0.69 + A(t_1) \cdot 1.13. \tag{7.11}$$

Notice that one can put $t_1 \approx t - d_2[1]$ for the A argument in Eq. (7.11) rather than solve Eq. (7.10) with respect to t_1. The new absolute value of d_3, $d_3 \to \tilde{\tilde{d_3}} \approx -\underline{77}$ is significantly larger than the value of this coefficient at the first step $\tilde{d_3} \approx \underline{14.7}$. At the same time, the first perturbation correction to Δ_1 in Eq. (7.11) looks admissible due to the moderate size of this term. Nevertheless, one can conclude that the next step of seBLM does not really improve the convergence of the perturbation series due to the large absolute value of $\tilde{\tilde{d_3}}$.

Finally, let us consider the results of the second stage of the seBLM procedure. Following Eq. (6.4), one finds that

1. $t_1 - t_2 = \Delta_{2,0} = d_3[1]/d_2[0] - 2d_2[1] \approx 166$ (!); therefore, $t_2 \approx t_1 - 166$ is outside the pQCD domain;

2. this scale-fixing procedure does not lead to a decrease of the $\tilde{\tilde{d_3}}$ term $\tilde{\tilde{d_3}} \to \tilde{\tilde{\tilde{d_3}}} = d_3[0] \approx -574$ due to a large value of the "genuine" term, compared to the contributions of the other terms in Eq. (7.4).

Here we encounter the particularly bad case when the second stage of the seBLM procedure cannot be applied even to the standard pQCD domain (see the first item above). But, if one, nevertheless, tries to apply it, this will not improve the convergence of the perturbation series (the second item) due to

$|d_3[0]| \gg d_3$. This happens because the procedure ignores the specific properties of the considered series, like alternate signs, etc. Let us recall that a similar effect at the LO was mentioned even in the pioneer work [1] in considering the branching of the Υ–decay, $\Gamma(\Upsilon \to \text{hadrons})/\Gamma(\Upsilon \to \mu^+\mu^-)$. The next section is devoted to the solution of this problem.

8. Generalized BLM procedure to improve the series convergence

How to improve the seBLM procedure. With hindsight, we now would be tempted to think that it would be better to have not performed the second stage at all and try instead another way to optimize the value of $\tilde{d}_3$ after the first one. Indeed, it is not mandatory to absorb the elements y_{ij} as a whole into the new scale following BLM, but take instead only that part which is appropriate for the coupling renormalization.[2] In our case it is tempting not to remove the contribution $y_{31} = d_3[2] + c_1 d_3[0,1]$ completely, as we did in Eq. (7.9) at the second step above, but rearrange a part of it and absorb one part into the coupling renormalization while keeping the other positive part in the remaining expression in order to compensate the large and negative $d_3[0]$ contribution. It turns out that it is convenient to introduce the x–portion of y_{31}, $x\ y_{31}$, in order to absorb it into the scale, see Eq. (8.2), while its $(1-x)$–portion, $(1-x)\ y_{31}$, is kept explicit in order to be able to cancel the negative "genuine" $d_3[0]$ in (8.1). This trick leads to a modified seBLM procedure termed x-dependent BLM, (xBLM),

$$\tilde{d}_3 \to \tilde{\tilde{d}}_3 = b_0^2 \cdot (1-x)(d_3[2] + c_1 d_3[0,1]) + b_0 \cdot (d_3[1] - 2d_2[0]d_2[1]) + d_3[0], \tag{8.1}$$

$$t - t_1 \ = \ \tilde{\Delta}_1 = d_2[1] + A(t_1) \cdot \left(x(d_3[2] + d_3[0,1]c_1) - d_2^2[1] - d_2[1]c_1\right). \tag{8.2}$$

Let us now establish an "optimization" condition, similar to Eq. (3.5), i.e., put $\tilde{\tilde{d}}_3 = 0$ to determine the value of x. One has to make sure that in the result the perturbative corrections are improved for both $\tilde{\tilde{d}}_3$ and $\tilde{\Delta}_1$ (see the fifth and sixth columns in Table 2) in comparison with those ones in Eqs. (7.9) and (7.11). The final result for D is then reduced to

$$D = 3\sum_f Q_f^2 \left\{1 + 3C_F \left[a(\tilde{t}_1) + \frac{1}{3} \cdot a^2(\tilde{t}_1) + 0 \cdot a^3(\tilde{t}_1)\right]\right\}, \tag{8.3}$$

As an illustration, the estimates of $\tilde{\Delta}_1(Q_i^2)$ at $Q_1^2 = 3$ GeV^2; $Q_2^2 = 26$ GeV^2 are shown in the sixth column of Table 2.

Table 2: Results of xBLM for $\tilde{\tilde{d}}_3 = 0$ ($\tilde{\tilde{r}}_3 = 0$) and $Q_1^2\ (s_1) = 3$ GeV^2; $Q_2^2\ (s_2) = 26$ GeV^2

N_f	b_0		x	$t - \tilde{t}_1 = \tilde{\Delta}_1$	$\tilde{\Delta}_1(Q^2)$		x	$t(s) - t(\tilde{s}_1) = \tilde{\Delta}_1$	$\tilde{\Delta}_1(s)$
3	9		0.56	$d_2[1] + a(\tilde{t}_1)b_0 \cdot 0.18$			1.84	$d_2[1] - a(\tilde{s}_1)b_0 \cdot 3.1$	
					$\tilde{\Delta}_1(Q_1^2)$				$\tilde{\Delta}_1(s_1)$
4	$\frac{25}{3}$	$\tilde{\tilde{d}}_3 = 0$	0.24	$d_2[1] - a(\tilde{t}_1)b_0 \cdot 0.45$	$\approx 0.58,$	$\tilde{\tilde{r}}_3 = 0$	2.56	$d_2[1] - a(\tilde{s}_1)b_0 \cdot 3.7$	≈ -0.27
					$\tilde{\Delta}_1(Q_2^2)$				$\tilde{\Delta}_1(s_2)$
5	$\frac{23}{3}$		-0.11	$d_2[1] - a(\tilde{t}_1)b_0 \cdot 1.19$	≈ 0.52		3.63	$d_2[1] - a(\tilde{s}_1)b_0 \cdot 4.48$	≈ -0.05

[2] This possibility has already been used for the LO BLM procedure, applied to the pion form factor in [29] and termed there $\overline{\text{BLM}}$, see p.13.

It is instructive to apply a similar procedure also to the observable quantity $R(s) = \dfrac{\sigma(e^+e^- \to h)}{\sigma(e^+e^- \to \mu^+\mu^-)}$ associated with D:

$$R(s) = D(s) - d_1 \frac{\pi^2}{3} \cdot b_0^2 \bar{a}^3 = 3 \sum_f Q_f^2 \left\{1 + 3C_F \left[a + r_2 a^2 + r_3 a^3\right]\right\}, \tag{8.4}$$

where $r_1 = d_1,\ r_2 = d_2,\ r_3 = d_3 - \dfrac{\pi^2}{3} \cdot b_0^2$ see, e.g., [26], $t(s) = \ln(s/\Lambda^2)$ and $a = \bar{a}(t(s))$. The large and negative π^2-term, arising due to the analytic continuation, makes r_3 also negative (cf. with Eq. (7.3)). As a result of the xBLM procedure, one should replace r_2 by $\tilde{d}_2$ and r_3 by $\tilde{r}_3$, whereas the x-dependent term in $\tilde{r}_3$ transforms to $b_0^2 \cdot (1-x)(d_3[2] + c_1 d_3[0,1] - \pi^2/3)$. Then, to obtain a positive compensating term to cancel $d_3[0]$ at the next step of the transformation (that leads to $\tilde{\tilde{r}}_3 = 0$), one should take $x > 1$, consult the eighth and ninth columns in Table 2. The result of this procedure is exemplified by showing partial values of shifts $\tilde{\Delta}_1(s_{1,2})$ that are placed in the last column of this table. For these examples, $\tilde{\Delta}_1 < 0$ and, therefore the optimal scale $\tilde{s}_1$ becomes larger than the initial one s.

Having developed the xBLM procedure, we are in the position to gain control over the size of the coefficients d_{i+2} and the magnitude of the associated scales t_i to balance their contributions. This possibility is of paramount importance because we need to control *several different* perturbation expansions in the generalized BLM procedure: one is the size of the expansion coefficients d_{i+2} like d_3 in Eq. (8.1), the others are for the scale shifts Δ_i like Δ_1 in Eq. (8.2). Finally, in the result of this procedure, Eqs. (8.3), (8.4), the coefficients of a^2 reduce to $d_2[0] = \dfrac{1}{3}$ (as for the standard BLM procedure), the coefficients $\tilde{\tilde{d}}_3$ ($\tilde{\tilde{r}}_3$) of a^3, reduce to 0, while the values of the new normalization scale $\tilde{t}_1$ ($t(\tilde{s}_1)$) are moderate and under control. This effect – to hold the control over both the types of expansions – is the main advantage of the xBLM procedure, while the condition $\tilde{\tilde{d}}_3 = 0$ used here is not a mandatory requirement for it. Instead of that one can require to significantly reduce the size of d_3 for better convergence. This trick to improve the convergence of the truncated series can be further generalized.

Matrix-form generalized BLM procedure. To generalize the seBLM procedure in the way mentioned above, let us introduce a lower triangular matrix $X = \{x_{ij}\},\ x_{ii} \equiv 0$ associated with the matrix Y instead of employing a single scalar parameter x. The element x_{ij} determines those portions of the contribution y_{ij} which should be absorbed into the coupling renormalization, according to the operation $x_{ij}\ y_{ij}$. On the other hand, the remainder of the contribution $y_{ij}\ \bar{x}_{ij}$ (where $\bar{x}_{ij} \equiv 1 - x_{ij}$) is kept in the PE-coefficient D_i. The choice $X = 0$ brings us back to the initial series before performing any transformations, while $X = \{x_{21} = 1,\ x_{i>j} = 0\}$ corresponds to the standard BLM procedure. The matrix $X = \{x_{i>j} = 1\}$ corresponds to the seBLM procedure, whereas all off-diagonal elements y_{ij} should be absorbed in new couplings. For the latter case, the first column $x_{i1} = 1$ itself leads to the first stage of seBLM, as one infers by comparing the second column in Table 3 with Eq. (B.1). For the discussed xBLM procedure the first column transforms into $\{1,\ A_1 y_{21}\bar{x}_{i1},\ A_1^2 y_{31}\bar{x}_{31},\ \ldots\}$ instead of the diagonalized structure $\{1,\ 0,\ 0,\ \ldots\}$ in seBLM. The schematic sketch of the first stage of xBLM is demonstrated in Table 3 and Eqs. (8.5)–(8.7) and one should compare them in conjunction with the seBLM procedure, presented in Table 1 (or in Eq. (B.1)).

$$\bar{A}(t) \to \bar{A}(t_1) \equiv A_1;\ t - t_1 \equiv \Delta_1 = \Delta_{1,0}(X) + A_1 \cdot \Delta_{1,1}(X) + \ldots. \tag{8.5}$$

The formulae for the Δ_1 proper scale, Eqs. (5.8), (5.9),... remain valid in this case by taking into account the obvious changes $y_{i1} \to y_{i1}x_{i1}$,

$$\Delta_{1,0}(X) = y_{21}x_{21}; \tag{8.6}$$

$$\Delta_{1,1}(X) = y_{31}x_{31} - 2(y_{21}x_{21})y_{21} + (y_{21}x_{21})^2 - (y_{21}x_{21})c_1. \tag{8.7}$$

Table 3: The structure of $A_1 \cdot A_1^{i-1} y_{ij}^{(1)}(X) b_0^{-j+1}$ at the first stage of the xBLM procedure

$A_1\cdot$	1			
$A_1\cdot$	$(A_1)^1\ y_{21}\bar{x}_{21}$	$+ A_1 \quad \frac{y_{22}}{b_0}$		
$A_1\cdot$	$(A_1)^2\ y_{31}\bar{x}_{31}$	$+ (A_1)^2 \frac{y_{22}}{b_0}\left(\frac{y_{32}}{y_{22}} - 2(y_{21}\ x_{21})\right)$		$+ (A_1)^2 \frac{y_{33}}{b_0^2}$
	$+\ldots$	$+\ldots$		$+\ldots$
$A_1\cdot$	$(A_1)^n\ y_{n1}\bar{x}_{n1}$	$+\ldots$		$+\ldots$

Additional free parameters x_{ij} give rise to a total amount of parameters $n(n-1)/2$ in N^{n}LO of the PE. This allows one to perform a "fine tuning" of the coefficients of the series minimizing those expansion coefficients pertaining to the considered order of the PE. A more complicated structure of the final PE series is the price one has to pay for such an improvement of the convergence of the series.

At the second stage of the xBLM procedure the contribution from the second column y_{i2} transforms into the sum $\sum (A_2)^{i-1}(y_{i2}\bar{x}_{i2})/b_0$ that should be compared with the single term $(A_2)^{i-1}\delta_{i2}\ y_{22}/b_0$ in the seBLM one. Here A_2 is fixed by the modified condition for Δ_2

$$\bar{A}(t_1) \to \bar{A}(t_2) \equiv A_2;\ t_1 - t_2 \equiv \Delta_2 = \Delta_{2,0}(X) + A_2 \cdot \Delta_{2,1}(X) + \ldots,$$
$$\Delta_{2,0}(X) = \frac{y_{32}}{y_{22}} x_{32} - 2(y_{21}\ x_{21}) \tag{8.8}$$

with still higher terms being omitted for simplicity and whereas the whole expression is similar to the one in Eq. (6.4). All other portions $y_{i2}\ x_{i2}$ of this column should be absorbed into the new coupling A_2 by means of the expansion coefficients $\Delta_{2,i}$. Further stages of the implementation of the xBLM procedure are similar to the first ones and result in

$$S(A) \to S_{x\mathrm{BLM}} = d_0 + \frac{A(t_1)}{b_0} \cdot d_1 \left[1 + \sum_{i \geq j \geq 2} \left(\bar{A}(t_j)\right)^{i-1} \bar{x}_{ij}\ y_{ij} \cdot b_0^{-j+1}\right]. \tag{8.9}$$

In Eq. (8.9) we lose the clear diagonal form, like Eq. (6.7), but obtain more flexibility to fix the values of the $A^i D_i$ terms in the expansion. The elements of the X matrix for various BLM procedures, are collected in Table 4.

Table 4: The stages of the generalization of the BLM procedure

"fine tuning" matrix	Standard PT	BLM	seBLM (first step)	seBLM	xBLM
X_{ij}	0	$\delta_{i2}\delta_{1j}$	$\theta(i>1)\ \delta_{ii}\delta_{1j}$	$\theta(i>j)\ \delta_{ii}\delta_{jj}$	$\theta(i>j)\ x_{ij}$
results for PT series	Eq. (2.1)	Eq. (3.6)	Eqs. (B.1), (5.1)	Eqs. (6.7), (6.8)	Table 2; Eq. (8.9)

Let us pause for a moment to make some clarifying remarks to a few partial cases: (i) The FAC setting in N^{2}LO corresponds to the condition $(A_1)^2\left(y_{21}\bar{x}_{21} + \frac{y_{22}}{b_0}\right) = 0$ which has been considered in

Eq. (3.5). (ii) On the other hand, if we restrict ourselves, say, to the N^{3}LO, then we have a matrix X with 3 parameters, x_{21}, x_{31}, x_{32} to optimize the contributions of A^2D_2 and A^3D_3, respectively,

$$A^2D_2 \to \; A_1\left[A_1 y_{21}\bar{x}_{21} + A_2\frac{y_{22}}{b_0}\right] = C_2; \tag{8.10}$$

$$A^3D_3 \to \; A_1\left[(A_1)^2 y_{31}\bar{x}_{31} + (A_2)^2\frac{y_{22}}{b_0}\left(\frac{y_{32}\;\bar{x}_{32}}{y_{22}}\right) + (A_2)^2\frac{y_{33}}{b_0^2}\right] = C_3; \tag{8.11}$$

$$\text{where} \qquad A_1 = \bar{A}(t-\Delta_1),\;\; A_2 = \bar{A}(t-\Delta_1-\Delta_2), \tag{8.12}$$

and $C_2,\; C_3$ are appropriate for us numbers, e.g., $C_2 = C_3 = 0$. The case discussed in Sec. 8 in conjunction with Table 2 corresponds to the partial solution of the above equations for $C_3 = 0$, $A_1 = A_2$ with $x_{21} = 1$, $x_{31} = x$, $x_{32} = 0$. A complete set of solutions to Eqs. (8.10)–(8.12) for $C_2 = C_3 = 0$ with respect to x_{ij} can be obtained and analyzed numerically.

9. Conclusions

In this paper we have considered the generalization of the BLM procedure with the aim to (i) extend it sequentially to any fixed order of the perturbative expansion in QCD and investigate the property of this extension, (ii) improve this generalization with respect to a better convergence of the perturbative expansion. To this end, a new hierarchy of the contributions to the expansion coefficients of a two-point function was designed which exploits the following properties of the beta-function coefficients: $b_i \sim b_0^{i+1}$. Employing this new hierarchy, we constructed a sequential and unambiguous generalization of the BLM procedure [1] to any fixed order of the perturbative expansion beyond the so-called "large-b_0" approximation, a procedure we called seBLM. This seBLM procedure leads to a new expansion series in terms of a set of new couplings $a_j = \bar{a}(t_j)$, Eq. (6.7), depending on a set of proper scales t_j that can also be rewritten in a form close to a continued-fraction representation, given by Eq. (6.8). The advantages and disadvantages of the seBLM procedure were exemplarily discussed in terms of the four–loop Adler D–function. We found that the seBLM procedure may still fail in improving the perturbation-series expansion for the D–function at its second stage, just for the same conditions under which also the standard BLM procedure fails. To improve the convergence of this series, the seBLM procedure was equipped by additional free parameters, encoded in a parameter x (see Eqs. (8.1)–(8.3) in Sec. 8) in the spirit of the Fast Apparent Convergence procedure [3]. These free parameters enable us to reduce the expansion coefficients of higher orders in a systematic way and at the same time to control the value of the couplings $\bar{a}(t_j)$. This improved procedure has been successfully applied to the series of the D– and $R_{(e^+e^-\to h)}$– functions with results presented in Table 2. The final generalization of the BLM procedure is provided by the xBLM one, which is now parameterized by a whole matrix $X = \{x_{ij}\}$ of additional parameters in order to achieve a better convergence of the perturbative expansion, as it was briefly discussed in the second part of Sec. 8. This way, one is enabled to control both the size of the expansion coefficients and the magnitude of the coupling a_j—and also of the associated scales t_j—from the most general point of view.

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A. Proof of the convolution representation

1. Let $\bar{A} \equiv \bar{A}_{(l)}$ be a solution of the RG equation with the l–loop B-function. Let us introduce the operator $\hat{D}_t = \dfrac{\bar{A}^2}{B(\bar{A})}\dfrac{d}{dt}$. Then, we find $\hat{D}_t \bar{A} = \bar{A}^2$ and

$$\frac{1}{n!}\left(\hat{D}_t\right)^n \ \bar{A} = \left(\bar{A}\right)^{n+1}. \tag{A.1}$$

Substituting (A.1) and Eq. (2.5) into Eq. (2.1), and collecting the corresponding terms in the exponent for, e.g., $D^+(n)$ part of the series,

$$\begin{aligned}\sum_{n=1} D^+(n)\bar{A}^n &= \int_0^\infty \mathcal{P}^+(\alpha)\left(\sum_{m=0}\frac{\left(\alpha\hat{D}_t\right)^m}{m!}\right)\ \bar{A}(t)d\alpha = \int_0^\infty \mathcal{P}^+(\alpha)\left[\exp\left(\alpha\hat{D}_t\right)\bar{A}(t)\right]d\alpha \\ &= \int_0^\infty \mathcal{P}^+(\alpha)\ \bar{A}\left(t+\alpha\cdot\frac{\bar{A}^2}{B(\bar{A})}\right)d\alpha,\end{aligned} \tag{A.2}$$

one arrives at expression (2.11). In the two-loop case the exact solution can be expressed in terms of the Lambert function $W(z)$, [30, 17] defined by

$$z = W(z)\exp\left(W(z)\right). \tag{A.3}$$

This solution has the explicit form

$$\bar{A}_{(2)}(t) = -\frac{1}{c_1}\frac{1}{1+W_{-1}(z_t)}, \tag{A.4}$$

where $z_t = (1/c_1)\exp\left[-t/c_1 - 1 + i\pi\right]$ and the branches of the multivalued function W are denoted by W_k, $k = 0, \pm 1, \ldots$. A review of the properties of this special function can be found in [30]; see also [17].

2. In the same way, one can obtain this representation for the *non-power* expansion series $\{A_n^{\rm APT}\}$ that appears instead of the standard powers $\{A^n\}$ in the analytic perturbation theory (APT) [23, 24]. The APT expansion of $S(A)$ in (2.1) can be written as

$$S(A) = d_0 + \frac{d_1}{b_0}\cdot\sum_{n=1} D_n\ A^n \to S^{\rm APT} = d_0 + \frac{d_1}{b_0}\cdot\sum_{n=1} D_n\ A_n^{\rm APT}. \tag{A.5}$$

The one-loop APT coupling constant is a bounded function of t, $A_1^{\rm APT}(t) = \dfrac{1}{t} - \dfrac{1}{e^t-1} \le 1$. The solutions for higher values of the index n of the one-loop constants $A_n^{\rm APT}$ are not powers of $A_1^{\rm APT}$, but they can be obtained from $A_1^{\rm APT}$ by means of the same relation expressed in Eq. (2.8), see [24], namely,

$$A_{n+1}^{\rm APT}(t) = \frac{1}{n!}\left(-\frac{d}{dt}\right)^n A_1^{\rm APT}(t) \neq (A_1^{\rm APT}(t))^{n+1}. \tag{A.6}$$

This leads to the same convolution structure given by Eq. (2.9) with $A_1^{\rm APT}(t)$ entering into the integrand. The finiteness of $A_1^{\rm APT}(t)$ guarantees the convergence of this integral representation. For this reason the corresponding series in APT, represented by Eq. (A.5), are really well-convergent [31].

3. The required β-function coefficients with the Minimal Supersymmetric Model (MSSM) light gluinos [32] are,

$$b_0\left(N_f,N_g\right) = \frac{11}{3}C_A - \frac{4}{3}\left(T_R N_f + \frac{N_g C_A}{2}\right); \tag{A.7}$$

$$b_1\left(N_f,N_g\right) = \frac{34}{3}C_A^2 - \frac{20}{3}C_A\left(T_R N_f + \frac{N_g C_A}{2}\right) - 4\left(T_R N_f C_{\rm F} + \frac{N_g C_A}{2}C_A\right); \tag{A.8}$$

$$\begin{aligned} b_2\left(N_f,N_g\right) = {} & \frac{2857}{54}C_A^3 - N_f T_R\left(\frac{1415}{27}C_A^2 + \frac{205}{9}C_A C_F - 2C_F^2\right) + \left(N_f T_R\right)^2\left(\frac{44}{9}C_F + \frac{158}{27}C_A\right) - \\ & \frac{988}{27}N_g C_A(C_A^2) + N_g C_A N_f T_R\left(\frac{22}{9}C_A C_F + \frac{224}{27}C_A^2\right) + \left(N_g C_A\right)^2\frac{145}{54}C_A\,. \end{aligned} \tag{A.9}$$

The b_3 coefficient, which includes the MSSM light gluinos, is not yet known, so we present it here in the standard [33] simplest form

$$\begin{aligned} b_3(N_f) = {} & \left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)N_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)N_f^2 + \\ & \frac{1093}{729}N_f^3\,. \end{aligned} \tag{A.10}$$

B. Explicit formulae for $y^{(1)}_{i,k}$ and $\Delta_{i,k}$

The explicit expressions for the few first elements of the matrix $Y^{(1)}$ are

$$\begin{aligned} & \bar{A}^1\ \cdot D_1 \to A_1^1\cdot\ \ 1; \\ & \bar{A}^2\ \cdot D_2 \to A_1^2\cdot\ \ 0 + \frac{y_{22}}{b_0}; \\ & \bar{A}^3\ \cdot D_3 \to A_1^3\cdot\ \ 0 + \frac{1}{b_0}\left(y_{32} - 2y_{21}\ y_{22}\right) + \frac{y_{33}}{b_0^2}; \\ & \bar{A}^4\ \cdot D_4 \to A_1^4\cdot\ \ 0 + \frac{1}{b_0}\left(y_{42} - 3y_{21}\ y_{32} + y_{22}[5y_{21}^2 - 2y_{31}]\right) + \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{1}{b_0^2}\left(y_{43} - 3y_{21}\ y_{33}\right) + \frac{y_{44}}{b_0^3}; \\ & A^n\ \cdot D_n \to A_1^n\cdot\ \ 0 + \frac{1}{b_0}\left(y_{n2} - \ldots\right)\ldots \qquad . \end{aligned} \tag{B.1}$$

The explicit expressions for the elements of the proper scales Δ_1 and Δ_2 are given by

$$\Delta_{1,0} = y_{21} = d_2[1]; \tag{B.2}$$

$$\begin{aligned} \Delta_{1,1} &= y_{31} - (y_{21})^2 - c_1 y_{21} = \\ &= d_3[2] - d_2^2[1] + c_1\left(d_3[0,1] - d_2[1]\right); \end{aligned} \tag{B.3}$$

$$\begin{aligned} \Delta_{1,2} &= y_{41} - 3y_{31}y_{21} - 2(y_{21})^3 - c_1 y_{31} + \frac{3}{2}c_1(y_{21})^2 + (c_1^2 - c_2)y_{21} = \\ &= d_4[3] - 3d_2[1]d_3[2] + 2(d_2[1])^3 + \\ &\quad c_1\left(d_4[1,1] - 3d_3[0,1]d_2[1] + \frac{3}{2}(d_2[1])^2 - d_3[2]\right) + \\ &\quad c_1^2\left(d_2[1] - d_3[0,1]\right) + c_2(d_4[0,0,1] - d_2[1]); \end{aligned} \tag{B.4}$$

$$\Delta_{2,0} = \frac{y_{32}}{y_{22}} - 2y_{21} = \frac{d_3[1]}{d_2[0]} - 2d_2[1]; \tag{B.5}$$
$$\Delta_{2,1} = \frac{y_{42}}{y_{22}} - 3y_{21}\frac{y_{32}}{y_{22}} + 5y_{21}^2 - 2y_{31} - \Delta_{2,0}^2 - c_1\Delta_{2,0};$$
$$\dots \ .$$

C. Distribution sense of sBLM.

It is instructive to illustrate the discussed seBLM procedure from the "distribution"-point of view, which is based on the convolution representation provided by Eq. (2.11). This has been demonstrated in Eq. (3.2) for the one-loop running coupling explicitly. One can re-derive the results of the seBLM procedure at the first stage using these terms, e.g., for the $\mathcal{P}^+$ case. This goes as follows. First, one expands the factor $\bar{A}_{(l)}$ in the integrand in Eq. (2.11) in the variable α. Second, following the line of argument underlying Sec. 5, one can arrive at the same Eqs. (5.8)–(5.10) for the elements. To achieve this goal, on should specify a partial generating function P_1 describing the distribution of the elements of the first column of the matrix Y, $\int_0^\infty P_1(\alpha)\alpha^{n-1}d\alpha \equiv \langle\alpha^{n-1}\rangle_1 = y_{n1}$, and then substitute P_1 in the place of $\mathcal{P}$ at the final stage of the procedure. As a result of the first seBLM stage, i.e., Eq. (5.11), the convolution $\langle\bar{A}_{(2)}(t-\dots)\rangle_1$ reduces to

$$A_1 \equiv \bar{A}(t-\Delta_1) = \int_0^\infty \delta(\alpha-\Delta_1)\bar{A}(t-\alpha)d\alpha \ .$$

Therefore, the distribution P_1 *reduces to* $\delta(\alpha-\Delta_1)$, an expression resembling the standard BLM result $\delta(\alpha-d_2[1])$ in Sec. 3. This term Δ_1 incorporates all the "α_s–renormalizations" associated with the single dressed-gluon propagator with two dressed vertices and the connected quark fields in the skeleton diagrams in Fig. 3(a). The common normalizing factor d_1 originates from the first tree diagrams of these skeleton Feynman graphs. At the second stage of the seBLM procedure we deal with the contribution from the second column of $Y^{(1)}$. Analogously, a normalized distribution P_2 can be introduced for $y_{n2}^{(1)}$, $\int_0^\infty P_2(\alpha)\alpha^{n-1}d\alpha \equiv \langle\alpha^{n-1}\rangle_2 = \tilde{y}_{n2}^{(1)}$. These contributions correspond to "two–gluon" skeleton diagrams illustrated in Fig. 3(b) (see also Fig. 2(c)). By now the effective scale of the coupling $\bar{A}(t-\Delta_1)$ is already fixed at the previous stage. As a result of this operation, $P_2 \to \delta(\alpha-\Delta_2)$ and the normalization scale of the couplings appear to be shifted, $\bar{A}(t-\Delta_1) \to \bar{A}(t-\Delta_1-\Delta_2)$. The normalizing factor $d_2[0]/b_0$ for that contribution originates from the undressed diagrams, which correspond to the skeleton graphs in Fig. 3(b).

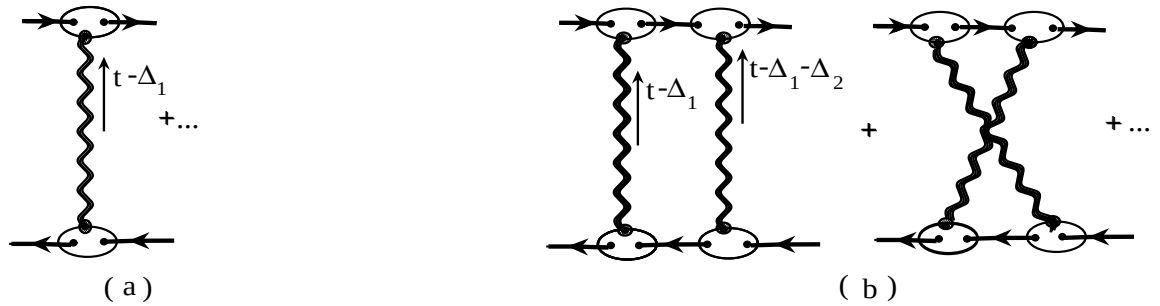

Figure 3: The elements of skeleton diagrams are presented: An oval box denotes a dressed vertex; a thick wavy line represents a dressed gluon propagator and a thick fermion line stands for dressed quark propagators; the arrow indicates the value of the seBLM "scale" of the corresponding effective charge. (a) Skeleton diagrams for the first stage of seBLM; (b) Skeleton diagrams with two dressed gluon lines for the second stage of seBLM.

Executing a number of seBLM stages for $\langle \bar{A}_{(2)}(t-\ldots)\rangle$, one arrives at the representation

$$\langle \bar{A}_{(2)}(t-\ldots)\rangle \to \int_0^\infty d\alpha \left(\delta(\alpha-\Delta_1)+\frac{d_2[0]}{b_0}\bar{A}(t-\Delta_1)\cdot\delta(\alpha-\Delta_1-\Delta_2)+\ldots\right)\bar{A}(t-\alpha). \qquad \text{(C.1)}$$

The kernel of this convolution can be compared with the PE representation in Eq. (3.7),

$$\mathcal{P}^+(\alpha)=\sum_n \frac{(-1)^n}{n!}\delta^{\{n\}}(\alpha)\cdot D^+(n) \to \delta(\alpha-\Delta_1)+d_2[0]/b_0\bar{A}(t-\Delta_1)\cdot\delta(\alpha-\Delta_1-\Delta_2)+\ldots \qquad \text{(C.2)}$$

The final seBLM series, Eq. (6.7), corresponds to the expansion encoded in the skeleton diagrams. The coefficients of this expansion $d_n[0]$ (y_{nn}) originate from the first undressed skeleton diagrams, whereas the scales Δ_n appear due to the renormalizations in these skeleton diagrams.

D. Representation for the D-function

The Adler D function is known [27] for the MSSM with N_g light gluinos, $D(a,N_f,N_g)$. On the other hand, one can obtain explicit expressions for the functions $N_f=N_f(b_0,b_1)$ and $N_g=N_g(b_0,b_1)$ solving the set of equations (A.7),(A.8) with respect to $N_f,\ N_g$. Substituting these solutions into $D(a,N_f,N_g)$, one arrives at the expansions Eq. (D.1), (D.2) and Eq. (D.4)–(D.8),

$$D(a)=3\sum_f Q_f^2\left\{d_0+d_1\left(a+d_2a^2+d_3a^3+\ldots\right)\right\},\ d_0=1;\ d_1=3C_F;$$

$$d_2=b_0d_2[1]+d_2[0]; \qquad \text{(D.1)}$$

$$d_3=b_0^2d_3[2]+b_1\cdot d_3[0,1]+b_0d_3[1]+d_3[0]; \qquad \text{(D.2)}$$

$$d_4=b_0^3d_4[3]+b_2\cdot d_4[0,0,1]+b_0b_1d_4[1,1]+b_0^2d_4[2]+b_1d_4[1]+b_0d_4[1]+d_4[0]\,. \qquad \text{(D.3)}$$

The N_f^2–terms of d_4 have recently been calculated in [28], but, unfortunately, these results cannot be used in our approach. The reason is that it is impossible to separate the terms $b_2d_4[0,0,1]$ and $b_0b_1d_4[1,1]$ that are of the order of $O(b_0^3)$ from the b_0^2–term, i.e., $b_0^2d_4[2]$, that also contributes to the "N_f^2 projection". This is why one needs a new degree of freedom (like the gluino).

$$d_2[1]=\frac{11}{2}-4\zeta_3\approx 0.691772; \qquad d_2[0] \;=\; \frac{C_A}{3}-\frac{C_F}{2}=\frac{1}{3}; \qquad \text{(D.4)}$$

$$d_3[2]=\frac{302}{9}-\frac{76}{3}\zeta_3\approx 3.10345; \qquad d_3[0,1]=\frac{101}{12}-8\zeta_3\approx -1.19979; \qquad \text{(D.5)}$$

$$d_3[1]=C_A\left(\frac{3}{4}+\frac{80}{3}\zeta_3-\frac{40}{3}\zeta_5\right)-C_F\left(18+52\zeta_3-80\zeta_5\right)\approx 55.7005; \qquad \text{(D.6)}$$

$$d_3[0]=\frac{1}{36}(523C_A^2+852C_AC_F-414C_F^2)-72C_A^2\zeta_3+\frac{5}{24}\left(\frac{176}{3}-128\zeta_3\right)\frac{(\sum_f Q_f)^2}{3(\sum_f Q_f^2)} \qquad \text{(D.7)}$$

$$\approx -573.9607-19.8326\frac{(\sum_f Q_f)^2}{3(\sum_f Q_f^2)}\,. \qquad \text{(D.8)}$$

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